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ASSET MANAGEMENT WITH TRADING UNCERTAINTY

Duncan K. Foley
Martin F. Hellwig

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massachusetts
institute of
technology
50 memorial drive
cambridge, mass. 02139

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I. INTRODUCTION

It is our purpose in this paper to explore the connections between uncertainties inherent in the trading process and the holding of liquid assets, with the idea that the techniques we develop here are a step toward a unified theory of money, price formation and the trading process.

What we call "trading uncertainty" is uncertainty about an agent's immediate opportunities to buy and sell. It is inherent in the operation of real markets when information cannot be transmitted or processed costlessly. Trading uncertainty must be sharply distinguished from the "state-of-the-world" uncertainty which is treated in the theories of portfolio choice and market allocation of risk.* State of the world uncertainty concerns events like changes

*Cf. Arrow, Debreu, ch. 7.

in endowments, technology and tastes that are generally taken as exogenous from the point of view of the economic relations of production and exchange. The major theme of the theory of state-of-the-world uncertainty is the ability of agents to reduce private riskiness by exchanging goods in different states of the world, that is, by insurance or hedging, to evade some or all of the possible

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risk.

The central fact about trading uncertainty is that it cannot be traded away, since it inheres in the trading process itself. It could be eliminated only in a world where the transmission and processing of information was costless.

The picture of market equilibrium of conventional theory explicitly rules out trading uncertainty since agents are supposed to be able to buy or sell any amount of any good at a known price. Trading uncertainty may enter in two guises depending on whether one views market disequilibrium in "Marshallian" or "Walrasian" terms. There may be several prices coexisting in the market at any moment, so that agents do not face a single known price. Or there may be only one price, but if it is not an equilibrium price agents cannot all buy or sell any amounts they want to. If excess supply or excess demand is rationed randomly in part, then agents will face trading uncertainty about their ability to buy or sell.

Costless and instantaneous transmission and processing of information would eliminate any divergence of price at a given moment, and would also presumably rule out trading at a disequilibrium price. This is the case studied by conventional theory (even with the introduction of state-of-the-world uncertainty). We believe it is a polar or asymptotic case to which real markets are only better or worse approximations.

The agents in real markets always bear some residual trading uncertainty. This uncertainty cannot be hedged or insured, since the insurance contract itself would be equivalent to the transaction it insures. Take, for instance, the case of an agent selling a house. Can the agent buy insurance as to the price he will get for the house? Clearly the issuer of such insurance is in

precisely the position of having bought the house at the insured price. The problems of consummating and insuring the transaction are economically equivalent.

The major theme we want to develop in this paper is that there is an intimate connection between trading uncertainty and the holding of liquid assets. We show in one particular model that without trading uncertainty agents will not hold liquid assets asymptotically while with trading uncertainty they will hold on average positive balances. This connection arises from stating the budget constraint in an intuitively appealing way: that the agent must not run out of stocks of purchasing power at any step of a sequential trading process.

This approach also offers an alternative and more tractable treatment of the idea advanced by Clower [1965, 1967] and Leijonhufvud [1967, 1968] that Keynes' consumption multiplier represents a recognition that spending patterns are constrained not by potential income at current market prices, but by realized income, which will be smaller than potential income whenever trading takes place at non-equilibrium prices. In our treatment, the inability of an agent instantaneously to sell at going market prices reduces his current consumption, but not in a mechanical one-to-one fashion. In addition, the risk of inability to sell affects the agent's consumption spending even in periods when he himself is lucky enough to achieve a sale at the market price. In models of the kind we study here there appears to be a precise and subtle account of the distinction between "notional" and "effective" demand, and of the connection between "effective" demand and holdings of asset balances.

II. A MODEL OF TRADING UNCERTAINTY

An economic agent exists in an infinite sequence of periods $t = 0, 1, 2, \ldots$ (We imagine these periods to be rather short in terms of calendar time, comparable to the time between transactions, on the order of hours or days.) He buys a consumption good, c, at a known money price p, which is assumed to remain constant, and sells labor, l, at a known money wage w, assumed also to remain constant. There is, however, a probability $(1-q)(0 \leq q \leq 1)$ that the agent will find himself unable to sell labor at all in the period, or equivalently, that in that period the particular agent (not necessarily every agent) will face a zero wage. The agent also owns a non-interest bearing asset, m, and cannot buy consumption without paying m for it. Since m is the only asset in the model and it is used for transactions we call it "money," although much of what we say could apply to assets in general as a store of wealth.

The agent maximizes the expected stream of discounted utility of consumption and labor. His decision variables are c_{1t}, c_{2t}, and l_t, where c_{1t} is his consumption purchase when he is employed, c_{2t} his consumption purchase when he is unemployed, and l_t his labor offering when he is employed. We can write this problem formally as:

$$(1) \qquad \text{Max} \quad E[\sum_{t=0}^{\infty} \alpha^t u(c_t, l_t)] \qquad (0 < \alpha < 1)$$

subject to

$$c_{1t}, c_{2t}, l_t \geq 0$$

$$\left.\begin{array}{l} c_t = c_{1t} \\ l_t = l_t \\ m_{t+1} = m_t + wl - pc_{1t} \end{array}\right\} \text{with probability } q \qquad \left.\begin{array}{l} c_t = c_{2t} \\ l_t = 0 \\ m_{t+1} = m_t - pc_{2t} \end{array}\right\} \text{with probability } (1-q)$$

$m_t \geq 0$ for all t

$m_0 \leq m$, given. $\quad (0 \leq q \leq 1)$

We make the following assumptions on the utility function:

a) The instantaneous utility function $u(c,l)$ is continuously differentiable, strictly concave and nonnegative.

b) For all c, l: $u_c > 0$, $u_l < 0$.

Furthermore,

$$\lim_{c \to 0} u_c(c,l) = \infty \quad \text{for all } l;$$

$$\lim_{l \to 0} u_l(c,l) = 0 \quad \text{for all } c;$$

and there is some $\bar{l}$ such that

$$\lim_{l \to \bar{l}} u_l(c,l) = -\infty \quad \text{for all } c.$$

Part b) ensures that the consumer never acquires an unlimited amount of money in any period.

We find it convenient to use the indirect instantaneous utility functions $v_1(\cdot)$, $v_2(\cdot)$, defined as:

$$v_1(e) = \max_{c,l \geq 0} u(c,l)$$

$$\text{subject to: } pc - wl \leq e$$

$$v_2(e) = \max_{c \geq 0} u(c,0)$$

$$\text{subject to: } pc \leq e.$$

$v_1(e)$ is the one period utility the agent can achieve by net dishoarding of e when he is employed; $v_2(e)$ the one period utility achieved by net dishoarding of e when he is unemployed. It is well known that under a), both $v_1(e)$ and $v_2(e)$ are strictly concave, nonnegative, increasing and continuously differentiable with $v_1(e)$ defined on $[-w\bar{l},\infty)$, $v_2(e)$ defined on $[0,\infty)$. Furthermore,

$$\lim_{e\to 0} v_2' = \infty \text{ and as a consequence } \lim_{e\to 0} v_2'(e) > \lim_{e\to 0} v_1'(e).$$

The value of the optimal program under (1) depends, for given prices p, w and probability of employment q, on initial money balances m only. It is natural to describe the agent's choice in terms of m_1, money balances held over into the next period when he is employed, and m_2, money balances held over when he is not employed. At any moment, the agent balances the marginal utility gained from spending his money immediately and the marginal utility of holding money into the next period. If we define the value of the program for initial balances m, as

$$V(m) = \max \ E \sum_{t=0}^{\infty} \alpha^t u(c_t, l_t)$$

etc., as in (1) then the latter component is given by the value, discounted by one period, of the whole program, starting with end-of-period money balances.

This line of reasoning leads to a dynamic programming formulation of the problem;* The function $V(\cdot)$ must satisfy the functional equation:

$$(2) \qquad V(m) = q \max_{m_1 \geq 0} [v_1(m-m_1) + \alpha V(m_1)]$$

$$+ (1-q) \max_{m_2 \geq 0} [v_2(m-m_2) + \alpha V(m_2)]$$

The total utility of money balances is given by taking the expectation from the employed and unemployed contingencies, where in each contingency utility is the sum of instantaneous utility from consuming and working in the current period and the discounted value of the program starting with m_1 (or m_2) in the next period.

In the mathematical appendix, we prove the following :

<u>Proposition 1:</u> An optimal policy exists to problem (1). The value of problem (1), $V(\cdot)$ is a strictly concave, strictly increasing, differentiable function of m that satisfies equation (2).

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*Cf. Levhari and Srinivasan.

III. PROPERTIES OF THE OPTIMAL POLICY

We continue to characterize the solution to the individual's problem in each period by m_1 and m_2, his end-of-period money balances. For given q, these depend on initial balances m and satisfy the first order conditions:

(3) $$v_1'(m-m_1) = \alpha V_m(m_1) + \mu_1$$

(4) $$v_2'(m-m_2) = \alpha V_m(m_2) + \mu_2$$

$$\mu_1, \mu_2 \geq 0; \; \mu_1 m_1 = \mu_2 m_2 = 0$$

Proposition 2: The policy correspondences $m_1(m,q)$, $m_2(m,q)$ are single-valued, continuous in m and satisfy for all $q < 1$:

a) $m_2(m) > 0$ for $m > 0$

b) $m_1(m) > 0$ for $m \geq 0$

c)* $0 < m_1'(m) < 1$ for $m \geq 0$

d)* $0 < m_2'(m) < 1$ for $m \geq 0$

e) For all q, there exists $m^*(q)$, such that $m_1(m^*(q)) = m^*(q)$.

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*Where m_1', m_2' do not exist, d) and e) are to be interpreted as statements about the right and left hand derivatives.

Proof: The fact that m_1, m_2 are single-valued follows from strict concavity of v_1, v_2, V, and the continuity of m_1, m_2 follows from the strict monotonicity and continuity of v_1', v_2', V_m.

a) By the envelope theorem

(5) $\quad V_m(m) = qv_1'(m-m_1) + (1-q)v_2'(m-m_2).$

As $m \to 0$, $m-m_2 \to 0$ and $v_2' \to \infty$.

Hence, for $q < 1$,

$$\lim_{m \to 0} V_m(m) = \infty$$

But then, $m_2(m) = 0$, $m > 0$ implies

$$v_2'(m-m_2(m)) < \alpha V_m(m_2(m)),$$

so that the first order condition (5) cannot be fulfilled. As long as there is a positive chance of being unemployed, it is worth holding some money over to avoid being caught without money or employment with an infinite marginal utility of consumption. The same argument establishes (b), with an extension to the case $m = 0$, since

$$\lim_{m \to 0} v_1'(m) < \infty.$$

From results a) and b), it follows that for $q < 1$, $\mu_1 = \mu_2 = 0$, for all m.

c) Differentiating (4), we get:

$$v_1'(m-m_1)(dm-dm_1) = \alpha V_{mm}(m_1)dm_1$$

or:

$$\frac{dm_1}{dm} = \frac{v_1''(m-m_1)}{v_1''(m-m_1)+\alpha V_{mm}(m_1)}$$

By strict concavity of v_1, V we have $v_1''<0$, $V_{mm}<0$ and hence,

$$0 < \frac{dm_1}{dm} < 1$$

Exactly the same argument establishes (d).

e) Suppose that there is a $\bar{q}$, such that, for this probability $\bar{q}$ of employment, and all m: $m_1(m)>m$. From the first order conditions, we know that then, for all m:

(6) $$v_1'(0) < \alpha V_m(m)$$

From the envelope theorem,

(7) $$\begin{aligned} V_m(m) &= \alpha q V_m(m_1(m)) + \alpha(1-q)V_m(m_2(m)) \\ &< \alpha V_m(m_2(m)), \end{aligned}$$

since $m_1(m) > m \geq m_2(m)$.

Writing $\tilde{m} = m_2(m)$, we have:

for all $\tilde{m}$:

$$V_m(\tilde{m}) > \alpha^{-1} V_m(m_2^{-1}(\tilde{m})).$$

Applying this relation recursively, we have, for all m, n:

$$V_m(m) > \alpha^{-n} V_m(m_2^{-n}(m))$$

But for all m, n, $m_2^{-n}(m)\varepsilon[0,\infty)$, so that

$$V_m(m_2^{-n}(m)) > \alpha^{-1} v_1'(0), \text{ by (6).}$$

By substitution we have for all m, n:

$$V_m(m) > \alpha^{-n-1} v_1'(0).$$

Hence, for all m:

$$V_m(m) = \infty, \text{ in contradiction to Proposition 1.} \quad \text{Q.E.D.}$$

Thus, we have proved the following: for a positive probability of unemployment, the agent will never run down his money balances to zero (a), (b); an increase in initial money balances will be used both to increase present net expenditure, and to increase end-of-period holdings (c), (d).

Of these results, only (e) carries over to the case where $q = 1$. Results (a) and (b) no longer hold, whereas (c) and (d) hold only where the first order conditions (3), (4) apply with $\mu_1 = \mu_2 = 0$; obviously $\mu_1 > 0$ or $\mu_2 > 0$ imply $m_1' = 0$, respectively $m_2' = 0$.

For $q = 1$, this implies:

$$V_m(m) = \alpha V_m(m_1(m)) + \mu_1.$$

Suppose that $m_1(m) > m$, for some $m > 0$. Since $\alpha < 1$ and V_m decreases in m, we must have $\mu_1 > 0$, therefore, $m_1(m) = 0$, a contradiction. Hence, $m_1(m) < m$, for all $m > 0$. By the monotonicity and nonnegativity of $m_1(m)$, we must have: $m_1(0) = 0$. But $v_1'(0) < \infty$ and hence, $V_m(0) < \infty$. Since $\alpha < 1$, we have for $m = 0$, $q = 1$,

$$\mu_1 = (1-\alpha)V_m(0) > 0.$$

This implies that the nonnegativity constraint on m_1 is strictly binding. Therefore, there exists a whole interval $I = [0,\varepsilon]$, $\varepsilon > 0$, such that $m \in I$ implies $m_1(m) = 0$.

IV: DYNAMIC ASPECTS OF THE MODEL

For initial money balances m, one begins the next period with money balances $m_1(m)$, if one is now employed, $m_2(m)$, if not. We may thus regard money balances at any moment in time as a random variable the distribution of which depends only on the probability of employment q and last period's money balances. The sequence $\{m_t\}_{t=0}^{\infty}$ is a Markov process given by the following rule:

$$m_{t+1} = \begin{cases} m_1(m_t) & \text{with probability } q \\ m_2(m_t) & \text{with probability } 1-q. \end{cases}$$

This process has the following properties:

Proposition 3: When $q = 1$, the agent reaches $m = 0$ in finite time and remains there forever.

Proof: From the closing paragraph of the preceding section we know that for $q = 1$, $m_{t+1} < m_t$ and that $m_t \epsilon [0,\varepsilon]$ implies $m_{t+1} = 0 \epsilon [0,\varepsilon]$.

Being monotone and bounded, the sequence m_t approaches a limit, say a. Suppose $a > 0$. But we also have $m_1(a) < a$. By continuity of $m_1(\cdot)$, there exists $\delta > 0$, such that $m_1(a+\delta) < a$. But for T large enough, $m_T < a + \delta$; therefore $m_{T+1} < a$, contradictory to the assumption that a is a limit to the monotone sequence $\{m_t\}_{t=0}^{\infty}$. Hence $a = 0$. But then, the interval $[0,\varepsilon]$ is reached in finite time, from which the proposition follows. Q.E.D.

It should be noted that in the other extreme case of $q = 0$, we have $\lim_{m \to 0} V_m(m) = \infty$ and therefore $m = 0$ implies $\mu_2 = 0$, so that the origin is only reached asymptotically.

The property that the probability distribution of money balances in the far future does not depend on initial money balances carries over to the general case: There is an ergodic distribution of money balances which is approached as t becomes large.

Proposition 4: For $0 \leq q \leq 1$ there exists a cumulative probability distribution $F^*(\cdot)$, defined on $[0,\infty)$ such that $\lim_{t\to 0} \text{Prob.}[m_t \leq m] = F^*(m)$ regardless of initial m. $F^*(\cdot)$ is the unique distribution function satisfying

$$(8) \quad F^*(m) = qF^*(m_1^{-1}(m)) + (1-q)F^*(m_2^{-1}(m)).$$

Proof: From Proposition 2, we know that for every q, there exists m^*, such that $m > m^*$ implies $m_1(m) < m$ and $m_2(m) < m$. All such m are therefore inessential. Once the agent reaches an $m > m^*$, it will never return to balances larger than m^*.

Hence, we only need concern ourselves with the interval $[0,m^*]$, for any given q, in studying the ergodic distribution.

Consider the space $\mathbb{P}$ of probability distributions on $[0,m^*]$, with the metric

$$\rho(F,G) = \int_0^{m^*} |F-G|$$

To see that this space is complete, consider any Cauchy sequence converging to a function F. We must show that F is a probability distribution.

By Helly's Compactness Theorem (Tucker, p. 83), such a sequence of probability distributions has a subsequence which converges pointwise to a distribution in $\mathbb{P}$, say G. Clearly, G is a limit of the subsequence for the metric ρ.

But every subsequence of our Cauchy sequence converges to the same limit F. Hence, $F = G$; therefore F is in $\mathbb{P}$ and $(\mathbb{P},\rho)$ is complete.

Next, consider the mapping $T:\mathbb{P}\to\mathbb{P}$ given by:

$$(TF)(m) = qF(m_1^{-1}(m)) + (1-q)F(m_2^{-1}(m)).$$

If for any period, F gives the distribution of money balances, then TF gives the distribution of money balances in the subsequent period. We show that T is a contraction mapping with respect to the metric ρ. For any two functions F, G in $\mathbb{P}$, we have

$$\rho(TF,TG) = \int_0^{m^*} |TF-TG| \leq q\int_0^{m^*}|F(m_1^{-1}(m)) - G(m_1^{-1}(m))|$$
$$+ (1-q)\int_0^{m^*} |F(m_2^{-1}(m)) - G(m_2^{-1}(m))|$$

For $0 \leq m < m_1(0)$, $m_1^{-1}(m)$ does not exist; we may write $F(m_1^{-1}(m) = G(m_1^{-1}(m)) = 0$. For $m_2(m^*) < m \leq m^*$, $m_2^{-1}(m)$ does not exist in $[0,m^*]$. However, if we imbed $\mathbb{P}$ in the space of probability distributions on $[0,\infty]$, $m_2^{-1}(m)$ exists in this interval, and, for F, G in $\mathbb{P}$, we have $F(m_2^{-1}(m)) = G(m_2^{-1}(m)) = 1$, where $m_2(m^*) < m \leq m^*$.

Hence, we may write:

$$\rho(TF,TG) \leq q \int_{m_1(0)}^{m^*} |F(m_1^{-1}(m)) - G(m_1^{-1}(m))|$$
$$+ (1-q)\int_0^{m_2(m^*)} |F(m_2^{-1}(m)) - G(m_2^{-1}(m))|$$

$$= q \int_0^{m^*} |F-G| m_1'(m) + (1-q) \int_0^{m^*} |F-G| m_2'(m)$$

$$\leq (1-c)\rho(F,G),$$

where $c > 0$ is chosen so that $1 - c \geq \max(m_1'(m), m_2'(m))$ for all $m \varepsilon [0,m^*]$. which is possible since m_1' and $m_2' < 1$, and $[0,m^*]$ is a compact interval.

Hence, T is a contraction mapping. By Banach's Fixed Point Theorem, (Kolmogorov-Fomin, p. 67), it has a unique fixed point and any sequence of functions $\{F_n\}$, where $F_n = T^n F_0$ approaches that fixed point irrespective of the initial function F_0. Q.E.D.

<u>Corollary</u>: The expected value of money balances under the ergodic distribution is given by the equation:

$$(9) \quad E(m|F^*) = qE(m_1(m)|F^*) + (1-q)E(m_2(m)|F^*)$$

<u>Proof</u>: By definition:

$$E(m|F^*) = \int_0^{m^*} m dF^*(m)$$

$$= q \int_0^{m^*} m dF^*(m_1^{-1}(m))$$

$$+ (1-q) \int_0^{m^*} m dF^*(m_2^{-1}(m))$$

$$= q \int_0^{m^*} m_1(m) dF^*(m)$$

$$+ (1-q) \int_0^{m^*} m_2(m) dF^*(m)$$

V. ECONOMIC BEHAVIOR AND THE PROBABILITY OF BEING EMPLOYED

In this section, we [first] analyze some effects of changes in q on the ergodic distribution. In general we will suppress the dependence of functions on q, but when necessary we write $m_1(m;q)$, $F^*(m;q)$ etc.

Proposition 5:

a) If there is certainty about the prospects for employment, the stationary distribution is concentrated at zero money balances:

$$F^*(0) = 1$$

b) If there is uncertainty about the prospects for employment, that is, if $0 < q < 1$, then money balances will almost never be run down to zero, that is:

$$F^*(0) = 0$$

Proof:

a) The case $q = 1$ has been dealt with already. In the case $q = 0$, we have:

$$F^*(m) = F^*(m_2^{-1}(m)).$$

But $m_2^{-1}(m) > m_1$ for $m > 0$ so that

$$F^*(m_2^{-1}(m)) \geq F^*(m).$$

Together with

$$\lim_{m\to\infty} F^*(m) = 1,$$

this gives $F^*(m) = 1$ for all $m > 0$. By continuity to the right of F^*, this implies $F^*(0) = 1$.

b) By Proposition 4, we have $F^*(0) = qF^*(m_1^{-1}(0)) + (1-q)F^*(m_2^{-1}(0))$. For $0 < q < 1$: $\{m_1^{-1}(0))\}$ is the null set, and $m_2^{-1}(0) = 0$. Hence, $F^*(0) = (1-q)F^*(0)$ so that $F^*(0) = 0$. Q.E.D.

It should be noted that the discontinuity of $F^*(0)$ in q at $q = 1$ and $q = 0$ does not arise from a discontinuity of $F^*(\ )$ at these points. In fact, as q approaches 1 or zero, the stationary distribution converges to the atomic distribution on $m = 0$.

This involves two things: First, probability mass becomes more and more concentrated close to the origin, i.e. high money holdings will become more and more improbable. Then, it also has to be shown that the limit of this process is the origin itself, that is to say that any positive holdings of money become improbable, if not excluded as q approaches either 1 or 0.

As q approaches 1, this is shown, if the essential interval itself shrinks and if m* approaches 0. In view of proposition 2, this is not trivial, since one has to show that $m_1(0;q)$ is continuous in q at the point $q = 1$, even though $V_m(0;q)$ is not.

At the other extreme, as q approaches 0, the situation is somewhat more difficult, because here, the essential interval will not vanish. Instead, we have to rely directly on the fact that it becomes less and less probable for money balances to increase over time. For this reason, proposition 6 will be somewhat weaker for the case $q \to 0$ than for $q \to 1$.

Proposition 6:

a) For any $m > 0$, there exists $\delta > 0$ such that $|1-q| < \delta$ implies $F^*(m;q) = 1$.

b) For any $m > 0$, $\varepsilon > 0$ there exists $\eta > 0$ such that $q < \eta$ implies $F^*(m;q) > 1-\varepsilon$.

Proof: We first show that for all m,

$$\lim_{q\to 1} m_1(m;q) \leq m,$$

with strict inequality holding for $m > 0$. From equation (7), we have as $q\to 1$:*

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*From Proposition 2(a,b), we know that $q < 1$ implies $m_1(m) > 0$, $m_2(m) \geq 0$ for all m, and hence $\lim_{q\to 1} \mu_1 = 0$ and $\lim_{q\to 1} \mu_2 = 0$.

$$\lim_{q\to 1} V_m(m;q) = \alpha \lim_{q\to 1} qV_m(m_1(m);q)$$

$$+ \alpha \lim_{q\to 1} (1-q)V_m(m_2(m);q)$$

The second term on the right hand side of this equation vanishes, unless $V_m(m_2(m);q)$ were to grow out of bounds as q approaches 1. This in turn would require

$$\lim_{q\to 1} V_m(0) = \infty$$

and

$$\lim_{q\to 1} m_2(m;q) = 0.$$

But then, for $m > 0$, $\alpha \lim_{q \to 1} V_m(0) > v_2'(m)$, in contradiction to the first order condition (4). Hence, we have, for $m > 0$,

$$\lim_{q \to 1} V_m(m;q) = \alpha \lim_{q \to 1} V_m(m_1(m);q)$$

Since $\alpha < 1$ and V_m decreases in m, we must have $\lim_{q \to 1} m_1(m) < m$ for all $m > 0$. From the monotonicity and nonnegativity of $m_1(\cdot)$, it follows that $\lim_{q \to 1} m_1(0) = 0$. The proposition follows immediately, because for q close enough to 1, $m_1(m;q) < m$, and m is an inessential state of the stochastic process so that $\bar{F}^*(m;q) = 1$.

b) From equation (8), we have,

$$\bar{F}^*(m) = q\bar{F}^*(m_1^{-1}(m)) + (1-q)\bar{F}^*(m_2^{-1}(m)) \geq (1-q)\bar{F}^*(m_2^{-1}(m))$$

Therefore, for all n,

$$\bar{F}^*(m) > (1-q)^n \bar{F}^*(m_2^{-n}(m)),$$

where $m_2^{-n}(\cdot)$ designates the n-fold inverse of the function $m_2(\cdot)$.

Also, from Proposition 2(e) we know that for all $q \in [0,1]$ there exists $m^*(q)$ with $m_1(m^*) = m^*$ and, from Proposition 4, $\bar{F}^*(m^*(q);q) = 1$.

In the appendix, we show that $m_1(\cdot)$ is continuous in q; therefore m^* is continuous in q and we can find max m^* as q varies over the compact interval $[0,1]$. Call this m^{**}. Also, define a function $f(\cdot)$, such that

$$f(m) = \max_q \; m_2(m;q).$$

Existence of this function follows from the continuity of $m_2(\cdot)$ in q, which again is shown in the appendix. Clearly, $f(\cdot)$ is continuous in m, with $0 < f' < 1$. For every m, then, we can find n such that $f^{-n}(m) \geq m^{**}$. By definition of the function f, we have for all $qm_2^{-n}(m) \geq f^{-n}(m) \geq m^{**}$. Also, by definition of m^{**}, $F^*(m^{**}) = 1$ for all q.

By monotonicity of F^*, we have for all q, $F^*(m_2^{-n}(m)) = 1$. Thus, for all m there exists n such that for all q,

$$F^*(m) \geq (1-q)^n.$$

Clearly, we can choose q small enough so that $F^*(m) \geq 1 - \varepsilon$. Q.E.D.

We have thus proved that not only does total certainty on the prospects of employment lead to zero money holdings in the long run, but that furthermore, in the proximity of the two certainty points, an increase in uncertainty leads to an increase in expected long run money holdings. It would be tempting to conclude that the maximum of expected long run money holdings as q varies concurs with the point of maximum uncertainty in the statistical sense, where $q = \frac{1}{2}$.

However, this is in general not true. The point of maximum expected long run money holdings will in general depend on the utility function, notably its third derivatives.

We therefore propose a different interpretation of our results in terms of willingness and ability to hold and acquire money. At the one extreme of certain employment, the agent is always able to acquire more money if only he wants to. But given his time preference and the fact that money earns him no interest, he never wants to and even runs down whatever balances he starts out with. At the other extreme of certain unemployment, the agent is always willing to acquire more money, if only he could do so. But he never is able to. Again time preference and the fact that he has no income whatsoever induce him to run down his initial balances.

It is clear that what we call "ability to acquire money" varies monotonically with the probability of employment: the more likely the agent is to be employed, the more chances he has of earning money, from which to increase his holdings.

As the probability of employment varies, variations in the willingness to hold money can be analysed either through variations in the marginal utility of money or through variations in the end-of-period money balances m_1 and m_2. To show that these variations are monotone in q, we formulate the following equivalent propositions:

Proposition 7: For all $m \in (0,m^*]$, both employed and unemployed money holdings decrease as q increases. The same holds for $m_1(0;q)$.

Proposition 8: For all $m \in (0, m^*]$ and all q_1, q_2, $q_1 < q_2$ implies

$$V_m(m;q_1) > V_m(m;q_2).$$

Proof: Equivalence of the two propositions follows immediately from the first order conditions (3) and (4). In fact, where the derivatives of m_1 and m_2 with respect to q exist, we have by total differentiation of (3) and (4):

$$\text{(10)} \qquad \frac{dm_1}{dq} = - \frac{\alpha V_{mq}(m_1;q)}{v_1'' + \alpha V_{mm}(m_1;q)}$$

$$\text{(11)} \qquad \frac{dm_2}{dq} = - \frac{\alpha V_{mq}(m_1;q)}{v_2'' + \alpha V_{mm}(m_2;q)}$$

Hence, it is sufficient to prove Proposition (8). From Lemma 5 in the appendix both m_1 and m_2 are continuous in q. Then V_m is continuous in q. For $m > 0$, V_m is of bounded variation as q varies over the compact interval [0,1]. Hence, V_m is almost everywhere differentiable with respect to q.*

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*Kolmogorov-Fomin, p. 331.

It is now sufficient to show that V_{mq} is negative wherever it exists.

Differentiating equation (7) with respect to q, we have:

$$\begin{aligned} V_{mq}(m;q) &= \alpha[V_m(m_1;q) - V_m(m_2;q)] \\ &+ \alpha[qV_{mq}(m_1;q) + (1-q)V_{mq}(m_2;q)] \\ &+ \alpha[qV_{mm}(m_1;q)\frac{dm_1}{dq} + (1-q)V_{mm}(m_2;q)\frac{dm_2}{dq}] \\ &= \alpha[V_m(m_1;q) - V_m(m_2;q)] \\ &+ \alpha qV_{mq}(m_1;q)\frac{v_1''(m-m_1)}{v_1''+\alpha V_{mm}(m_1;q)} \\ &+ \alpha(1-q)V_{mq}(m_2;q)\frac{v_2''(m-m_2)}{v_2''+\alpha V_{mm}(m_2;q)} \end{aligned}$$

For given q, consider the supremum of V_{mq} over the interval$[0, m^*]$. Suppressing q in the arguments of the functions, we have:

$$\begin{aligned} \sup_{[0,m^*]} V_{mq}(m) &= \alpha[V_m(m_1(\bar{m})) - V_m(m_2(\bar{m}))] + \alpha[qV_{mq}(m_1(\bar{m}))\frac{dm_1}{dm} + (1-q)V_{mq}(m_2(\bar{m})\frac{dm_2}{dm}] \\ &\leq \alpha[V_m(m_1(\bar{m})) - V_m(m_2(\bar{m}))] + \alpha[\sup_{[0,m^*]} V_{mq}(m)][q\frac{dm_1}{dm} + (1-q)\frac{dm_2}{dm}] \end{aligned}$$

where $\bar{m}$ is the value of m, for which V_{mq} achieves its supremum on$[0, m^*]$.

Now we have:

$$(12)\quad \sup_{[0,m^*]} V_{mq}(m) \leq [V_m(m_1(\bar{m})) - V_m(m_2(\bar{m}))] \frac{\alpha}{1 - \alpha(q \frac{dm_1}{dm} + (1-q) \frac{dm_2}{dm})}$$

But, on the interval $[0,m^*]$, we have $m_1 \geq m \geq m_2$, so that the right hand side of this inequality is always negative . Hence, on the interval $[0,m^*]$, V_{mq} is always negative. Q.E.D.

It should be noted that the restriction to the essential interval is not always needed. For the use of the supremum above, any closed interval containing the essential interval would do. The use of m* was important only in that it ensured the negativity of the right hand side of (12). Any other condition which ensures that $m_1 > m_2$ would serve the same purpose. For instance, if we could show that for all e $v_1'(e) < v_2'(e)$, then Proposition 7 will hold for all m. Unfortunately this does not appear to be true for all utility functions.

VI. THE CONSUMPTION FUNCTION AND LIQUIDITY PREFERENCE

The consumption function and its corollary the multiplier are central puzzles in the problem of relating macroeconomic ideas to microeconomics. Why should current income constrain current spending in an economy where agents command liquid assets? Both the life-cycle hypothesis of Modigliani and his associates and Friedman's permanent income hypothesis suggest that current income should affect spending only to the extent that it affects lifetime resources. Furthermore, to the extent that a change in current income is anticipated, it will not affect spending plans at all. On the other hand, Clower suggests that the consumption function has to be regarded as a short term liquidity phenomenon, the theoretical treatment of which requires a fundamental separation of earning and spending decisions.

The present model begins to illuminate this puzzle as well as the relation between consumption and the demand for liquid assets through its treatment of the budget constraint for sequential trading.

The preceding propositions establish two separate influences of unemployment on consumption, for given money balances m. First, the unemployed agent will consume $c_2(m)$ rather than $c_1(m)$. This effect corresponds in spirit to Clower's distinction between notional and effective demand. Notice that neither $c_1(m)$ nor $c_2(m)$ is really "notional" demand in the sense of the demand the agent would have if trading uncertainty did not exist at all and he could always sell as much labor as he wanted at the going rate. In that kind of world $q = 1$ and money balances would in the long run play no part. Notice, too that $c_2(m) < c_1(m)$ as long as consumption is not an inferior good but not by any mechanical application of a "money constraint." There is only one constraint,

that the agent never run out of money. Still, the fact that c_2 is smaller than c_1 does reflect the agent's response to his immediate failure to sell labor in the light of the necessity to use money as a means of payment, which is close in spirit to Clower's idea.

Second, since net spending during the period will increase with the probability of employment q, both in the employed and the unemployed case, if we assume that neither consumption nor leisure are inferior goods, it follows that in all cases consumption will increase with the probability of employment. This change in spending due to a change in the expected degree of trading uncertainty, alters the agent's estimate of his total lifetime resources. It is thus akin to a change in expected permanent income.*

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*The analytical relationship between permanent income and q is somewhat cumbersome: Since there is no discounting in the model, which would make earlier time periods more important, we may evaluate expected permanent income from the ergodic distribution:

$$y_p = qw \int l(m,q)\,dF^*(m)$$

The assertion that permanent income is monotone in q is equivalent to the assertion that effects of q on the willingness to work when employed and on the ergodic distribution do not overcompensate for the direct effect of q on y_p. In the simple case, where $L(m,q) = \bar{l}$, we have: $y_p = qw\bar{l}$.

Both these effects work in the same direction. On the other hand, we have seen that the corresponding effects on money held over into the next period work in opposite directions. If widespread unemployment occurs without any change in expectations about individual probabilities of unemployment, then agents will dishoard to maintain their consumption (Proposition 2). If subjective estimates of the chance of unemployment increase, then agents will hoard to accu-

mulate money (Proposition 5). For any short period of time, this provides an observable distinction between short term liquidity and long term behavioral aspects of consumption theory. Unfortunately, if the expectation of unemployment is linked to the actual experience, then over any longer period, this distinction disappears because the change of expected money holdings is indeterminate. It is probably this indeterminancy of the long run relationship between expected money holdings and expected consumption, which has diverted attention away from the interdependence of the consumption function and money demand (liquidity preference). However, the preceding analysis makes clear that it is not the difference between notional and effective demand that relates to liquidity preference but rather the impact of different expectations on long run behavior.

It may be argued that since our model only has one asset, it is a model of saving rather than of preference for liquid assets. However, the motive for saving is precautionary and thereby closely related to the motive for holding liquid assets. As in the life-cycle model, the consumer saves when he is employed for periods when he earns nothing. But whereas in the life-cycle model, the consumer knows the exact timing of periods in which he earns and periods in which he does not earn, in this model he knows nothing about the timing and only a probabilistic statement about the overall incidence of non-earning periods is possible. There is no return to holding the asset and, in the absence of uncertainty, asset holdings will be zero in the long run. In this sense we think that the type of saving that occurs in this model may be considered as a case of liquidity preference.

VII. AGGREGATE DEMAND AND MARKET INFORMATION

We have seen that unemployment affects the demand for consumption both through a short run liquidity and through a long run behavioral effect. It has been suggested by Leijonhufvud and others that these effects impair the ability of the markets to adjust from a disequilibrium position. Within the static context of our model, we want to make this notion more explicit by considering the behavior of market aggregates.

Suppose that there are two types of agents, with a continuum of each. Each type supplies one good inelastically and buys the other good. Market conditions are characterized by four variables, namely, the prices of labor and the consumption good, w and p, and the probabilities that a supplier is able to sell labor or the consumption good at the given market price, q and s respectively.

Furthermore, suppose that money balances for agents of type one are distributed according to the distribution $F_1(\ )$. Then, the total demand for consumption goods is:

$$\int_0^{m_1^*} c(m,q)\,dF_1(m)$$

where $c(m,q)$ is the consumption demand for money balances m and employment prospects q.

Likewise, for type two, the demand for labor will be:

$$\int_0^{m_2^*} l(m,s)\,dF_2(m)$$

where $l(m,s)$ is the demand for labor under money balances m and sales prospects s, $F_2(\ )$ is the distribution of money balances for type two. If the actual incidence of unemployment or inability to sell consumption goods does not dis-

tinguish between different agents of a given type and corresponds to q and s, then money balances in the subsequent period are distributed as

$$F_1^{t+1} = qF_1^t(m_{11}^{-1}(m;q)) + (1-q)F_1^t(m_{12}^{-1}(m;q))$$

and likewise for type two, where the first subscript distinguishes the type of consumer, and the second the contingency.

We see that this transformation is identical to the transformation T used in Section IV to describe the consumer's money balances $t + 1$ periods from the initial point of decision making. But where in that case, F was a distribution of money balances as expected for some future period, in the present context, it describes the actual distribution over different agents of one type.

If F_1 happens to be the ergodic distribution for type one, then money balances for type one in subsequent periods are identically distributed. Hence, as a group, agents of type one are in this case neither accumulating nor decumulating money, although as individuals, their balances are constantly changing. In this case, the value of their consumption spending as a group will be equal to their achieved labor sales:

$$\int_0^{m_1^*} c(m;q)dF_1^*(m) = qw\bar{l}/p$$

Similarly, for type two:

$$\int_0^{m_2^*} l(m;s)dF_2^*(m) = sp\bar{c}/w$$

If achieved sales by one type are to be equal to effective demand by the other type, we get the pseudo-equilibrium condition:

$$qw\bar{l} = sp\bar{c}$$

For such a pseudo-equilibrium, the total demand for money is given as:

$$M = \int_0^{m_1^*} m\,dF_1^*(m) + \int_0^{m_2^*} m\,dF_2^*(m).$$

For any choice of q, s there exist p, w such that at these prices and sales prospects the market is in a pseudo-equilibrium, and aggregate demand for any commodity is equal to expected achievable sales. Different choices of q,s result in different holdings of money balances.

There is nothing in the formalism of this model to suggest that the competitive equilibrium with $q = s = 1$ and zero money balances plays any special part. In fact, it is seen that any situation in which $q = s$ has a pseudo-equilibrium at the same relative price as the competitive equilibrium. This formalizes Leijonhufvud's notion that the reduction of effective demand by laid-off workers may be just large enough to justify the reduced production by firms and to prevent a return to equilibrium.

Of course, these considerations are subject to severe qualifications.*

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*A minor problem is presented by the assumption that firms produce on order and make the production decision under certainty about effective demand by workers who, on their side, are supposed to know already whether they are employed or not. We think that this problem can be amended by a richer dynamic structure, which includes firms which make production decisions before knowing what their sales are going to be and which hold inventories.

This model says nothing about how prices are set and employment or sales prospects are generated. Without an explicit dynamic structure, which gives an economic explanation for the setting of prices, no detailed conclusions can be drawn from the above model. However, if we assume that the change of money

prices in any market will be of the same sign as the effective excess demand, then the above model shows that there can be situations in which both the money wage rate and the consumption goods money price decrease. A priori there is then no presumption in favor of the argument that the real price moves closer to its equilibrium value. In fact, one could imagine processes in which for a given pair q,s and real wage w/p, the real wage remains constant, even through both w and p decrease. In this situation, the only adjustment towards equilibrium would be provided by a real balance effect, arising from the change in the demand for money as both p and w decline.

Another phenomenon worth considering is the role of expectations in this model. We mentioned before that the employment (sales) prospects as arguments of the behavioral functions are to be seen as expectations. In the simple model above, we see that such expectations are, in general, self-fulfilling in that a decreased expectation of future employment will decrease demand for consumption goods and thereby induce producers to employ fewer people, so that employment prospects are in fact reduced. This is similar to the phenomenon of self-fulfilling expectations in the theory of speculation, where the expectation by a sufficiently large group of agents that a price will rise will in fact cause that price to rise. In this model all agents in one market have the same expectations about that market. Therefore, it appears desirable to include in a dynamic model an account of how expectations are generated and to what extent they will be independent for different agents of a given type.

VIII. CONCLUSION

The present model has certain formal similarities to recent work in stochastic growth theory. Iwai and Brock and Mirman analyze optimal growth of an economy in which production in each period depends not only on capital and labor inputs, but also on a random factor. They show that the distribution of per capital capital and output approaches a stationary distribution, which may be interpreted as the equivalent of the consumption turnpikes in optimal growth under certainty.

The present paper differs from their work in two respects. On the one hand, the proof of ergodicity of the distribution of money balances is simpler in that it does not make use of kernels, but uses a contraction mapping argument. More important is the fact that if we interpret the structure of our model in the context of price uncertainty or production uncertainty, then we consider two-point distributions, where one point occurs at the origin. Under the assumption that $\lim_{c \to 0} u_c = \infty$, this class of distributions allows more specific propositions about optimal behavior. In that sense, we believe that trading uncertainty, although it has formal similarities to a special case of price or production uncertainty, merits a separate treatment, which we believe will throw new light on certain problems of disequilibrium adjustment.*

*The same could be said in a comparison of this model with the work of Merton who introduces uncertainty about wage income into his model of consumption and portfolio selection and analyzes the general analogy between wage income uncertainty and asset return uncertainty.

The most important aspect of this line of reasoning lies, in our opinion,

more in the imagery of its description of economic activity than in its mathematics. We can summarize the crucial elements as follows.

First, there is a residual and unhedgeable uncertainty as to an agent's opportunities to buy and sell. Second, the agent confronts these opportunities in a sequence of decisions, not all at once at some initial trading session. Third, the agent sees its budget constraint not in terms of current expenditures equalling current income nor as the present value of lifetime expenditures equalling the present value of lifetime income, but as never being able to incur negative money balances.

The conventional theory of consumer choice bears to this theory the same kind of relation that the theory of a "perfect" gas bears to that of actual gases. The conventional theory is an asymptotic or polar case in which trading uncertainty is ignored. It can be derived from the present theory by setting $q = 1$, that is, by eliminating trading uncertainty. Notice that the introduction of trading uncertainty implies certain qualitative changes in behavior (willingness to hold money, for instance) which do not disappear even when randomness itself has been eliminated by aggregation.

The present work is clearly only a first step in applying the three principles we have just mentioned to problems of economic theory. It should be possible to include more than one asset, by assuming that the alternative asset to money is not perfectly liquid in that it cannot always be sold at a known money price, just as labor cannot always be sold in the present model. An attractive feature of the present model that we believe will carry over to generalizations is that the distribution of assets arises in a natural way in the course of explaining aggregate phenomena. The relation between distributions of money and propositions reminiscent of Say's Law or Walras' Law seems

particularly interesting.

At the present stage of development this line of research offers no complete solutions to the pressing problems of general equilibrium theory, particularly the need to explain the apparently connected phenomena of price formation, money holding, and unemployment of resources in the framework of general equilibrium. The present work does touch on the last two of three topics from the point of view of the individual agent. What is missing is any consideration of decentralized price formation in a sequential trading framework. It seems very desirable to us that solutions to these problems reflect one of the properties of the present model, that is, that the conventional theory be derivable from the general theory as an asymptotic case when some cost goes to zero.

APPENDIX: EXISTENCE OF THE OPTIMAL POLICY

In this appendix, we prove Proposition 1 by induction on the sequence of finite horizon problems with the same constraints and objective function.

Consider the sequence of functions:

$$V^0(m) = qv_1(m) + (1-q)v_2(m)$$

$$V^T(m) = q \max_{m+w1 \geq m_1^T \geq 0} [v_1(m-m_1^T) + \alpha V^{T-1}(m_1^T)]$$

$$+ (1-q) \max_{m \geq m_2 \geq 0} [v_2(m-m_2^T) + \alpha V^{T-1}(m_2^T)]$$

By inspection V^T is the value and m_1^T, m_2^T the optimal policies of the T-period finite horizon problem of the same form as (1).

<u>Lemma 1</u>: Under assumptions for every T a unique V^T exists satisfying (3), which is strictly increasing, strictly concave, and differentiable in m. The derivative $V_m^T(m) = qv_1'(m-m_1^T(m)) + (1-q)v_2'(m-m_2^T(m))$.

<u>Proof</u>: V^0 clearly satisfies all the claims of the Lemma by assumption A. To make a proof by mathematical induction, assume that V^{T-1} satisfies all the claims.

$V^T(m)$ exists and is unique since it is the maximum value of a continuous function over a compact set.

$V^T(m)$ is strictly increasing because v_1, v_2 are strictly increasing, so a strictly higher value can be achieved by spending more in the first period in each contingency, even holding the rest of the program constant.

$V^T(m)$ is strictly concave since it is the sum of maxima of strictly concave functions over convex sets.

The m_1^T, m_2^T satisfy the first order conditions

$$v_1'(m-m_1^T) = \alpha V_m^{T-1}(m_1^T) + \mu_1 \qquad \mu_1 m_1 = 0 \qquad \mu_1 \geq 0$$

$$v_2'(m-m_2^T) = \alpha V_m^{T-1}(m_2^T) + \mu_2 \qquad \mu_2 m_2 = 0 \qquad \mu_2 \geq 0$$

Since v_1, v_2 and V^{T-1} are everywhere differentiable and strictly concave the m_1^T, m_2^T functions are unique and continuous in m.

We can write

$$\begin{aligned} V^T(m+h) - V^T(m) = q[&v_1(m+h-m_1(m+h)) + \alpha V^{T-1}(m_1(m+h)) \\ &- v_1(m-m_1(m)) - \alpha V^{T-1}(m_1(m))] \\ &+ (1-q)[\ldots]. \end{aligned}$$

Then, working only with v_1, since v_2 is similar :

$$\begin{aligned} &v_1(m+h-m_1(m+h)) + \alpha V^{T-1}(m_1(m+h)) - v_1(m-m_1(m)) - \alpha V^{T-1}(m_1(m)) \\ &= v_1'(\hat{e})[h - (m_1(m+h) - m_1(m))] + \alpha[m_1(m+h) - m_1(m)]V_m^{T-1}(m_1^*) \\ &= v_1'(\hat{e})h + [m_1(m+h) - m_1(m)][\alpha V_m^{T-1}(m_1^*) - v_1'(\hat{e})] \end{aligned}$$

by the mean value theorem, where $m_1^* \in [m_1(m+h), m_1(m)]$ and $\hat{e} \in [m+h-m_1(m+h), m-m_1(m)]$.

As $h \to 0$, $(m_1(m+h) - m_1(m))/h$ is bounded between the right and left hand derivatives of $m_1(m)$, and by the first order conditions together with continuity of m_1, m_2, $\alpha V^{T-1}(m_1^*) - v_1'(\hat{e})$ approaches zero. Therefore,

$$\lim_{h \to 0} \frac{V^T(m+h) - V^T(m)}{h} = qv_1'(m-m_1^T(m)) + (1-q)v_2'(m-m_2^T(m))$$

Q.E.D.

Now we consider the convergence of the V^T and V_m^T functions as T becomes large.

Lemma 2: Let $m_1^T(\cdot)$ and $m_2^T(\cdot)$ be the policy functions in the T-period problem. Then $m_1^T(m) \geq m_1^{T-1}(m)$ and $m_2^T(m) \geq m_2^{T-1}(m)$ for all m.

Proof: The first order conditions, and lemma 1 give, for all T,

$$v_1'(m-m_1^T) = \alpha V_m(m_1^T) = \alpha[qv_1'(m_1^T - m_1^{T-1}(m_1^T)) + (1-q)v_2'(m_1^T - m_2^{T-1}(m_1^T))$$

$$v_2'(m-m_2^T) = \alpha V_m(m_2^T) = \alpha[qv_1'(m_2^T - m_1^{T-1}(m_2^T)) + (1-q)v_2'(m_2^T - m_2^{T-1}(m_2^T))].$$

If $m_1^T(\hat{m}) < m_1^{T-1}(\hat{m})$ for some $\hat{m}$, it means that the agent spends more in the employed contingency when he has T periods to face than when he has T-1 periods. The corresponding $v_1'(m-m_1^T) < v_1'(m-m_1^{T-1})$, and so in at least one of

the second period contingencies his marginal utility must also be lower, meaning that in that contingency he spends more when he has T-1 periods to face than when he has T-2. Thus there is at least one chain of contingencies starting from $\hat{m}$ where the agent spends strictly more at each step when he is facing T periods than he did when he faced T-1. But this is impossible because under the T-1 period program he ended the last period with zero money balances in every chain of contingencies, and a policy of spending strictly more at each step would imply negative money balances at the period T-1 in that chain, which is not permitted. Therefore, $m_1^T(m) \geq m_1^{T-1}(m)$ and, by a parallel argument, $m_2^T(m) \geq m_2^{T-1}(m)$ for all m. Q.E.D.

<u>Lemma 3</u>: On any compact interval $[\underline{m},\overline{m}]$ the functions m_1^T, m_2^T converge uniformly to limiting functions m_1, m_2.

<u>Proof</u>: For all T, $m_1^T \leq m + w\overline{1} \leq \overline{m} + w\overline{1}$

$$m_2^T \leq m \leq \overline{m}$$

Hence, the functions m_1^T, m_2^T are uniformly bounded.

Furthermore, from the first order conditions, we have for all T m^*, $\tilde{m}$

$$|m^*-\tilde{m}| < \varepsilon \quad \text{implies} \quad |m_1^T(m^*) - m_1^T(\tilde{m})| < \varepsilon$$
$$\text{and} \quad |m_2^T(m^*) - m_2^T(\tilde{m})| < \varepsilon$$

Hence, the m_1^T and m_2^T are equicontinuous in m. It follows by Arzela's Theorem (kolmogorov-Fomin, p. 102) and the monotonicity of the sequences shown in Lemma 2 that the functions m_1^T and m_2^T converge uniformly to limiting functions m_1, m_2.

<u>Lemma 4</u>: The sequence of functions $V^T(\cdot)$ converges to a unique, differentiable, function $V(\cdot)$. The derivative $V_m(\cdot)$ is the limit of the derivatives V_m^T.

Proof: We appeal to the fact that if the derivatives of a sequence of functions are uniformly convergent on an interval, then the sequence converges uniformly to a limit function on the interval and the derivative of the limit function will exist and be equal to the limit of the derivatives of the original sequence. [Cf. Apostol, Mathematical Analysis, p. 402.]

By Lemma 1,

$$V_m^T(m) = qv_1'(m-m_1^T(m)) + (1-q)v_2'(m-m_2^T(m)).$$

Since v_1' and v_2' are continuous, and m_1^T, m_2^T converge, it is clear that $V_m^T(m)$ converges to some value V_m.

To see that this convergence is uniform on any compact interval, excluding the origin, consider that v_1' and v_2', since they are continuous, are uniformly continuous on a compact interval excluding the origin. For any ε we can choose a single δ such that

$$|v_1'(m-m_1(m)) - v_1'(m-m_1^T(m))| < \frac{\epsilon}{q} \qquad \text{and}$$

$$|v_2'(m-m_2(m)) - v_2'(m-m_2^T(m))| < \frac{\epsilon}{(1-q)}$$

whenever $|m_1(m) - m_1^T(m)| < \delta$ and $|m_2(m) - m_2^T(m)| < \delta$. But the uniform convergence of m_1^T and m_2^T assure that there is a single t^* so that for all $T > t^*$ these last conditions will be met.

Proposition 1: An optimal policy exists to problem (1). The value of problem (1), V( ) is a strictly concave, strictly increasing, differentiable function of m that satisfies equation (2).

The limit policy $(m_1(\cdot), m_2(\cdot))$ is clearly feasible for the program (1).

Is there any better policy? Suppose that there were, and consider that the maximum utility achievable in the T^{th} period starting with balances m is $v_1(m + Tw\bar{1})$, which is the amount the agent would have if he worked the maximum amount in every period. The tail of any policy is thus worth less than $\alpha^T \sum_{t=0} \alpha^t v_1(m+(T+t)w\bar{1})$ which goes to zero as T becomes large because $v_1(\cdot)$ is a concave function. Thus any policy that is strictly better for an infinite program than the limit policy would also yield more utility in the first T periods for some T than the m_1^T, m_2^T policy, which is a contradiction. Therefore $(m_1(\cdot), m_2(\cdot))$ is the optimal policy, and $V(\cdot)$ satisfies equation (2).

Since $V(\cdot)$ is the limit of a sequence of strictly concave, strictly increasing functions, $V(\cdot)$ must be concave and non-decreasing. Since $V(\cdot)$ satisfies equation (2) by the maximum principle and v_1, v_2 are strictly concave and strictly increasing it is clear that $V(\cdot)$ will also be strictly concave and strictly increasing. Q.E.D.

<u>Lemma 5:</u> The functions $m_1(m;q)$ and $m_2(m;q)$ are jointly continuous in (m,q).

<u>Proof:</u> First we prove by induction that m_1^T, m_2^T, V^T are jointly continuous in (m,q).

$$m_1^0(m;q) = m_2^0(m;q) = 0, \text{ and}$$

$$V^0(m;q) = qv_1(m) + (1-q)v_2(m),$$

so that m_1^0, m_2^0 and V^0 are jointly continuous in (m,q).

Assume m_1^{T-1}, m_2^{T-2}, V^{T-1} are jointly continuous in (m,q). Then m_1^T is the solution to

$$\max_{0\leq m_1^T\leq m+w\bar{1}} [v_1(m-m_1^T) + \alpha V^{T-1}(m_1;q)], \text{ and similarly for } m_2^T.$$

In Lemma 6, below, we show that a unique maximizer is jointly continuous in parameters of the objective function whenever the objective function is jointly continuous in the parameters and the maximizing variable. m_1^T and m_2^T are unique by concavity of v_1, v_2 and V^{T-1}, so this Lemma applies.

Second, we show that V^T converges uniformly over any compact set of (m,q) to the limit function V. In the proof of Proposition I we show that

$$|V^T(m;q) - V(m;q)| < \alpha^T \sum_{t=0}^{\infty} \alpha^t v_1(\bar{m} + (T+t)w\bar{1})$$

(where $\bar{m}$ is an upper bound on m), which can be made as small as we like by choosing T large independent of m and q. This establishes uniform convergence of the V^T sequence, and as a consequence that V is jointly continuous in m and q.

Finally, note that m_1 and m_2 are unique maximizers of a jointly continuous function, so that Lemma 6 establishes their joint continuity in m and q. Q.E.D.

<u>Lemma 6</u>: Let $X(\alpha)$ be the set of maximizers over a compact set S of a continuous function $f(x;\alpha)$ defined over a set $S\times T$. Then $X(\alpha)$ is upper-semi-continuous. If the correspondence $X(\alpha)$ is a function, then it is continuous in α.

Proof: Consider a sequence $\{\alpha^i\} \to \alpha^*$ and a convergent subsequence $\{x^i\} \to x^*$ where $x^i \epsilon X(\alpha^i)$. Suppose $x^* \notin X(\alpha^*)$. Then there exists $\bar{x} \epsilon S$ with

$$f(\bar{x};\alpha^*) > f(x^*;\alpha^*).$$

Then $f(\bar{x};\alpha^*) - f(x^*;\alpha^*) > 2\epsilon > 0$.

For $i > I_1$, we have

$$|f(\bar{x};\alpha^i) - f(\bar{x};\alpha^*)| < \epsilon \quad \text{and}$$

for $i > I_2$ we have

$|f(x^i;\alpha^i) - f(x^*;\alpha^*)| < \epsilon$ by continuity of the function $f(\cdot)$. Then for all $i > \max\,[I_1, I_2]$

$f(\bar{x};\alpha^i) > f(x^i;\alpha^i)$. But this contradicts $f(\bar{x};\alpha^i) \leq f(x^i;\alpha^i)$ by construction of the sequence x^i. Thus no such $\bar{x}$ can exist, and $x^* \epsilon X(\alpha^*)$. Q.E.D.

BIBLIOGRAPHY

1. Apostal, T.M., Mathematical Analysis, Addison-Wesley, Reading, 1957.

2. Arrow, K.J., "The Role of Securities in the Optimal Allocation of Risk-Bearing," Review of Economic Studies 31 (April, 1964), pp. 91-96.

3. Brock, W.A. and L.J. Mirman, "Optimal Economic Growth and Uncertainty: The Discounted Case," Journal of Economic Theory 4 (June, 1972), pp. 479-513.

4. Clower, R.W., "A Reconsideration of the Microfoundations of Monetary Theory," Western Economic Journal 6 (1967), pp. 1-9.

5. ________, "The Keynesian Counter-Revolution: A Theoretical Appraisal," in The Theory of Interest Rates, ed. by F.H. Hahn and F. Brechling, New York: Macmillan, 1965, ch. 5, pp. 103-25.

6. Debreu, G., Theory of Value, New York: Wiley, 1959, ch. 7.

7. Friedman, M., A Theory of the Consumption Function, Princeton: Princeton University Press, 1957.

8. Iwai, K., A Theory of Optimal Capital Accumulation Under Uncertainty, Part II, unpublished Doctoral Dissertation, M.I.T., 1972.

9. Kolmogorov, A.N. and S.V. Fomin, Introductory Real Analysis, Englewood Cliffs: Prentice Hall, 1970.

10. Leijonhufvud, A., "Keynes and the Keynesians: A Suggested Interpretation," American Economic Review 57:2 (1967), pp. 401-410.

11. ________, Keynesian Economics and the Economics of Keynes, Oxford University Press, 1968.

12. Levhari, D. and T.N. Srinivasan, "Optimal Savings Under Uncertainty," Review of Economic Studies (1969), pp. 153-164.